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Combalgebraic structures on decorated cliques

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Abstract. A new hierarchy of combinatorial operads is introduced, involving families of regular polygons with configurations of arcs, called decorated cliques. This hierarchy contains, among others, operads on noncrossing configurations, Motzkin objects, forests, dissections of polygons, and involutions. All this is a consequence of the definition of a general functorial construction from unitary magmas to operads. We study some of its main properties and show that this construction includes the operad of bicolored noncrossing configurations and the operads of simple and double multi-tildes. We focus in more details on a suboperad of noncrossing decorated cliques by computing its presentation, its Koszul dual, and showing that it is a Koszul operad.

Keywords: Operad; Koszul duality; Graph; Triangulation; Noncrossing configuration.

Introduction

Regular polygons endowed with configurations of arcs are very classical combinatorial objects. Up to some restrictions or enrichments, these polygons can be put in bijection with several combinatorial families. Triangulations are the most celebrated among these, but also noncrossing configurations [6], dissections of polygons, noncrossing partitions, or involutions belong also to this world. As many combinatorial objects, the polygons of most of these families can be described by composing or grafting smaller pieces together. Operads [10, 12] are algebraic structures abstracting the notion of planar rooted trees and their grafting operations. For this reason, operads are one of the most suitable modern algebraic structures to study such objects. In the last years, a lot of combinatorial sets have been endowed fruitfully with a structure of an operad (see for instance [3, 11, 4, 8, 7]), each time providing results about enumeration, discovering new statistics, or establishing new links (by morphisms) between different combinatorial sets.

The purpose of this work is twofold. First, we are concerned in endowing the whole set of polygons with configurations of arcs with a structure of an operad. This leads to see these objects under a new light, stressing some of their combinatorial and algebraic properties. Second, we would provide a general construction of operads of polygons rich enough so that it includes some already known operads. As a consequence, we obtain

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alternative constructions of existing operads and new interpretations of these. We work here with $\mathcal{M}$-decorated cliques, that are complete graphs whose arcs are labeled on $\mathcal{M}$, where $\mathcal{M}$ is a unitary magma. These objects are natural generalizations of polygons with configurations of arcs since the arcs of any $\mathcal{M}$-decorated clique labeled by the unit of $\mathcal{M}$ are considered as missing. The elements of $\mathcal{M}$ different from the unit allow moreover to handle polygons with arcs of different colors. We propose a functor C from the category of unitary magmas to the category of operads. It builds, from any unitary magma $\mathcal{M}$, an operad $\mathrm{C}\mathcal{M}$ on $\mathcal{M}$-decorated cliques.

This operad has a lot combinatorial and algebraic properties. First, $\mathrm{C}\mathcal{M}$ admits as quotients of operads several structures on particular families of polygons with configurations of arcs. We can for instance control the degrees of the vertices, the crossings, or the nestings between the arcs to obtain new operads. We hence get quotients of $\mathrm{C}\mathcal{M}$ involving, among others, Schröder trees, dissections of polygons, Motzkin objects, forests, with colored versions for each of these. This leads to a new hierarchy of operads, wherein links between its elements appear as surjective or injective morphisms of operads (see Figure 1). One of the most notable of these is built by considering the $\mathcal{M}$-

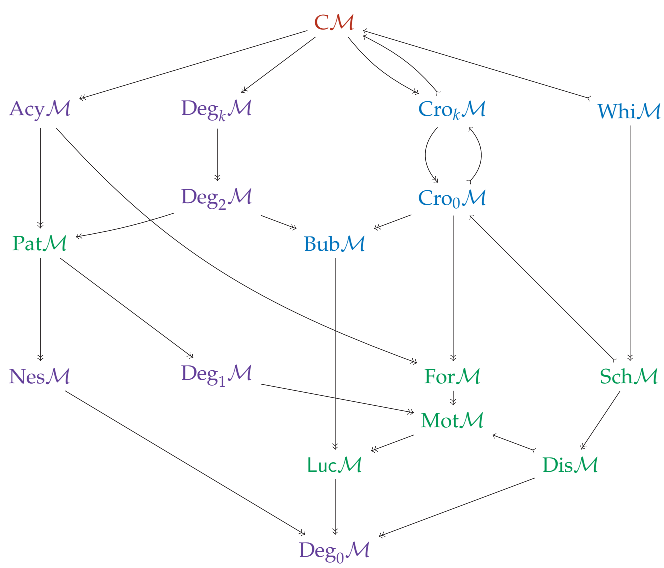


Figure 1: A diagram of suboperads and quotients of $\mathrm{C}\mathcal{M}$. Arrows $\rightarrowtail$ (respectively $\twoheadrightarrow$) are injective (respectively surjective) morphisms of operads. Here, $\mathcal{M}$ is a unitary magma without nontrivial unit divisors.

decorated cliques that have vertices of degrees at most 1, leading to a quotient $\mathrm{Inv}\mathcal{M}$ of $\mathrm{C}\mathcal{M}$ involving standard Young tableaux (or equivalently, involutions). To the best of our knowledge, $\mathrm{Inv}\mathcal{M}$ is the first nontrivial operad on these objects. Besides, the construction C allows to retrieve the operad BNC of bicolored noncrossing configurations [4] and the operads MT and DMT respectively defined in [11] and [8] that involve multi-tildes and double multi-tildes, operators coming from formal languages theory [2]. The sub-operad $\mathrm{NC}\mathcal{M}$ of $\mathrm{C}\mathcal{M}$ of $\mathcal{M}$-noncrossing configurations, that are $\mathcal{M}$-decorated cliques without crossing diagonals, admits some nice algebraic properties. It is first generated by elements of arity two (which is not the case of $\mathrm{C}\mathcal{M}$), and its nontrivial relations are concentrated in arity three. This operad is also a Koszul operad.

This text is organized as follows. The construction C is defined in Section 1 and its first properties are listed. Among other, we describe the generators of $\mathrm{C}\mathcal{M}$, its dimensions, establish that it admits a cyclic operad structure, and define two alternative bases, the H-basis and the K-basis, by considering a partial order structure on the set of $\mathcal{M}$-decorated cliques. Section 2 is devoted to define some quotients of $\mathrm{C}\mathcal{M}$ and to construct, through C, the operads BNC, MT, and DMT. Finally, we study in more details the operad $\mathrm{NC}\mathcal{M}$ in Section 3. We show that this operad is an operad of Schröder trees with labels on arcs satisfying some conditions. We compute its dimensions, its presentation, its Koszul dual, and establish the fact that it is a Koszul operad.

Notations and general conventions. All the algebraic structures of this article have a field of characteristic zero $\mathbb{K}$ as ground field. We shall use the classical notations about operads [10] and more precisely those of [7]. Since we consider only nonsymmetric operads, we call these simply *operads*. The sequences of integers cited in the sequel come from [13].

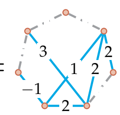
1 Operads of decorated cliques

1.1 Unitary magmas and decorated cliques

A *clique* of *size* $n \geqslant 1$ is a complete graph $\mathfrak{p}$ on the set of vertices $[n+1]$. An *arc* of $\mathfrak{p}$ is a pair of integers (x, y) with $1 \leqslant x < y \leqslant n+1$, a *diagonal* is an arc (x, y) different from $(x, x+1)$ and $(1, n+1)$, and an *edge* is an arc of the form $(x, x+1)$ and different from $(1, n+1)$. We denote by $\mathcal{A}_{\mathfrak{p}}$ the set of arcs of $\mathfrak{p}$. The *i-th edge* of $\mathfrak{p}$ is the edge $(i, i+1)$ and the arc $(1, n+1)$ is the *base* of $\mathfrak{p}$. Let $\mathcal{M}$ be a unitary magma, that is a set endowed with a binary operation $\star$ admitting a left and right unit $\mathbb{1}_{\mathcal{M}}$. An *$\mathcal{M}$-decorated clique* (or an *$\mathcal{M}$-clique* for short) is a clique $\mathfrak{p}$ endowed with a map $\phi_{\mathfrak{p}} : \mathcal{A}_{\mathfrak{p}} \to \mathcal{M}$. For convenience, for any arc (x, y) of $\mathfrak{p}$, we shall denote by $\mathfrak{p}(x, y)$ the value $\phi_{\mathfrak{p}}((x, y))$. Moreover, we say that the arc (x, y) is *labeled* by $\mathfrak{p}(x, y)$. When the arc (x, y) is labeled by an element different from $\mathbb{1}_{\mathcal{M}}$, we say that the arc (x, y) is *solid*.

In our graphical representations, we shall stick to the following drawing conventions for $\mathcal{M}$-cliques. First, each $\mathcal{M}$-clique is depicted so that its base is the bottommost segment and vertices are implicitly numbered from 1 to $n+1$ in the clockwise direction. Second, the label of any arc (x, y) of $\mathfrak{p}$ is represented in the following way. If (x, y) is solid, we represent it by a line decorated by $\mathfrak{p}(x,y)$. If (x,y) is not solid and is an edge or the base of $\mathfrak{p}$, we represent it as a dashed line. In the remaining case, when (x,y) is a diagonal of $\mathfrak{p}$ and is not solid, we do not draw it.

To explore some examples in this article, we shall consider the additive unitary magma $\mathbb{Z}$, the cyclic additive unitary magma $\mathbb{N}_\ell$ on $\mathbb{Z}/_{\ell\mathbb{Z}}$, and the unitary magma $\mathbb{D}_\ell$ on the set $\{\mathbb{1}, 0, a_1, \dots, a_\ell\}$ where $\mathbb{1}$ is the unit of $\mathbb{D}_\ell$, 0 is absorbing, and $a_i \star a_j = 0$ for all $i, j \in [\ell]$. For instance,

$\mathfrak{p} :=$  (1.1.1)

is a $\mathbb{Z}$-clique of size 6 such that, among others, $\mathfrak{p}(1,2) = -1$, $\mathfrak{p}(1,5) = 1$, $\mathfrak{p}(3,7) = 3$, $\mathfrak{p}(5,7) = 2$, $\mathfrak{p}(2,3) = 0$, and $\mathfrak{p}(2,6) = 0$.

1.2 A functor from unitary magmas to operads

For any unitary magma $\mathcal{M}$, we define the vector space $\mathrm{C}\mathcal{M} := \bigoplus_{n \geqslant 1} \mathrm{C}\mathcal{M}(n)$ where $\mathrm{C}\mathcal{M}(1)$ is the linear span of the singleton consisting in the $\mathcal{M}$-clique of size 1 whose base is labeled by $\mathbb{1}_\mathcal{M}$, and for any $n \geqslant 2$, $\mathrm{C}\mathcal{M}(n)$ is the linear span of all $\mathcal{M}$-cliques of size n. We endow $\mathrm{C}\mathcal{M}$ with a partial composition map $\circ_i$ defined linearly in the following way. If $\mathfrak{p}$ and $\mathfrak{q}$ are two $\mathcal{M}$-cliques of respective sizes n and m, and i is a valid integer, $\mathfrak{p} \circ_i \mathfrak{q}$ is obtained by gluing the base of $\mathfrak{q}$ onto the i-th edge of $\mathfrak{p}$, by relabeling the common arcs between $\mathfrak{p}$ and $\mathfrak{q}$, respectively the arcs $(i, i+1)$ and $(1, m+1)$, by $\mathfrak{p}(i, i+1) \star \mathfrak{q}(1, m+1)$, and by renumbering the vertices of the clique thus obtained from 1 to $n+m-1$ (see Figure 2). For example, in $\mathrm{C}\mathbb{Z}$, one has the two partial compositions

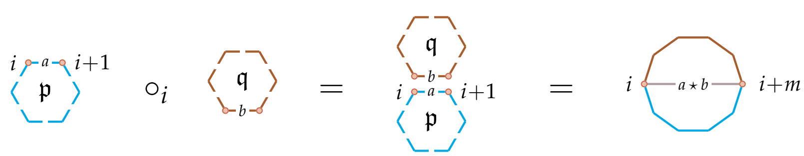


Figure 2: The partial composition of $\mathrm{C}\mathcal{M}$. Here, $\mathfrak{p}$ and $\mathfrak{q}$ are two $\mathcal{M}$-cliques. The label of the i-th edge of $\mathfrak{p}$ is $a \in \mathcal{M}$ and the label of the base of $\mathfrak{q}$ is $b \in \mathcal{M}$. The size of $\mathfrak{q}$ is m.

 (1.2.1)

Moreover, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two unitary magmas and $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a unitary magma morphism, we define $\mathrm{C}\phi : \mathrm{C}\mathcal{M}_1 \to \mathrm{C}\mathcal{M}_2$ as the linear map sending any $\mathcal{M}_1$-clique $\mathfrak{p}$ of size n to the $\mathcal{M}_2$-clique $(\mathrm{C}\phi)(\mathfrak{p})$ of size n such that, for any arc $(x,y) \in \mathcal{A}_\mathfrak{p}$, $((\mathrm{C}\phi)(\mathfrak{p}))(x,y) := \phi(\mathfrak{p}(x,y))$.

Theorem 1.2.1. *The construction* C *is a functor from the category of unitary magmas to the category of operads. Moreover,* C *respects injections and surjections, and all operads of the image of* C *are set-operads.*

Proof. We just sketch the proof of the fact that $\mathsf{C}\mathcal{M}$ is an operad when $\mathcal{M}$ is a unitary magma. This amounts to prove that the partial composition of $\mathsf{C}\mathcal{M}$ satisfies, for all $\mathcal{M}$-cliques $\mathfrak{p}$ (respectively $\mathfrak{q}$, $\mathfrak{r}$) of sizes n (respectively m, k), $(\mathfrak{p} \circ_i \mathfrak{q}) \circ_{i+j-1} \mathfrak{r} = \mathfrak{p} \circ_i (\mathfrak{q} \circ_j \mathfrak{r})$ where $i \in [n]$, $j \in [m]$, $(\mathfrak{p} \circ_i \mathfrak{q}) \circ_{j+m-1} \mathfrak{r} = (\mathfrak{p} \circ_j \mathfrak{r}) \circ_i \mathfrak{q}$ where $i < j \in [n]$, and $\circ_1 \mathfrak{p} = \mathfrak{p} = \mathfrak{p} \circ_i$ where $i \in [n]$. Each of these relations can be checked for example with the help of Figure 2. $\square$

1.3 General properties

Proposition 1.3.1. *Let* $\mathcal{M}$ *be a finite unitary magma. For all* $n \geqslant 2$, $\dim \mathsf{C}\mathcal{M}(n) = m^{\binom{n+1}{2}}$, *where* $m := \#\mathcal{M}$.

If $\mathfrak{p}$ is an $\mathcal{M}$-clique, we say that two diagonals (x, y) and (x', y') of $\mathfrak{p}$ are *crossed* if $x < x' < y < y'$ or $x' < x < y' < y$. Let $\mathfrak{G}_{\mathsf{C}\mathcal{M}}$ be the set of all $\mathcal{M}$-cliques $\mathfrak{p}$ such that, for any diagonal (x, y) of $\mathfrak{p}$, there is at least one **solid** diagonal (x', y') of $\mathfrak{p}$ such that (x, y) and (x', y') are crossed. Observe that, according to this description, all $\mathcal{M}$-cliques of size 2 belong to $\mathfrak{G}_{\mathsf{C}\mathcal{M}}$.

Proposition 1.3.2. *Let* $\mathcal{M}$ *be a unitary magma. The set* $\mathfrak{G}_{\mathsf{C}\mathcal{M}}$ *is the unique minimal generating set of* $\mathsf{C}\mathcal{M}$.

Recall that an operad $\mathcal{O}$ defined in the category of sets is *basic* [14] if all the maps $\circ_i^y : \mathcal{O}(n) \to \mathcal{O}(n + |y| - 1)$, $y \in \mathcal{O}$, defined by $\circ_i(x) := x \circ_i y$ are injective.

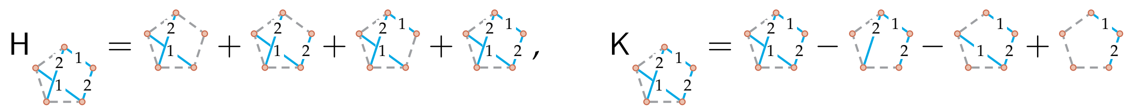
Proposition 1.3.3. *Let* $\mathcal{M}$ *be a unitary magma. As a set-operad,* $\mathsf{C}\mathcal{M}$ *is basic if and only if* $\mathcal{M}$ *is right cancellable.*

Let $\rho : \mathsf{C}\mathcal{M} \to \mathsf{C}\mathcal{M}$ be the linear map sending any $\mathcal{M}$-clique $\mathfrak{p}$ to the $\mathcal{M}$-clique obtained by rotating by one step $\mathfrak{p}$ in the counterclockwise direction.

Proposition 1.3.4. *Let* $\mathcal{M}$ *be a unitary magma. The map* ρ *endows* $\mathsf{C}\mathcal{M}$ *with a cyclic operad structure.*

Let $\preceq_{\mathrm{be}}$ (respectively $\preceq_{\mathrm{d}}$) be the partial order relation on the set of all $\mathcal{M}$-cliques, where, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$, one has $\mathfrak{p} \preceq_{\mathrm{be}} \mathfrak{q}$ (respectively $\mathfrak{p} \preceq_{\mathrm{d}} \mathfrak{q}$) if $\mathfrak{q}$ can be obtained from $\mathfrak{p}$ by replacing some labels $\mathbb{1}_{\mathcal{M}}$ of its edges or its base (respectively solely of its diagonals) by other labels of $\mathcal{M}$. For any $\mathcal{M}$-clique $\mathfrak{p}$, let the elements of $\mathsf{C}\mathcal{M}$ defined by $\mathsf{H}_{\mathfrak{p}} := \sum_{\mathfrak{p}' \preceq_{\mathrm{be}} \mathfrak{p}} \mathfrak{p}'$ and $\mathsf{K}_{\mathfrak{p}} := \sum_{\mathfrak{p}' \preceq_{\mathrm{d}} \mathfrak{p}} (-1)^{\mathrm{ham}(\mathfrak{p}', \mathfrak{p})} \mathfrak{p}'$, where $\mathrm{ham}(\mathfrak{p}', \mathfrak{p})$ is the *Hamming distance* between $\mathfrak{p}'$ and $\mathfrak{p}$, that is the number of arcs (x, y) such that $\mathfrak{p}'(x, y) \neq$

$\mathfrak{p}(x, y)$. By triangularity and by Möbius inversion, the family of all the $\mathsf{H}_{\mathfrak{p}}$ (respectively $\mathsf{K}_{\mathfrak{p}}$) forms a basis of $\mathbf{C}\mathcal{M}$, called H-*basis* (respectively K-*basis*). For instance, in $\mathbf{C}\mathbb{Z}$,


(1.3.1)

If $\mathfrak{p}$ is an $\mathcal{M}$-clique, we denote by $\mathfrak{p}_0$ (respectively $\mathfrak{p}_i$) the label of its base (respectively i-th edge). Moreover, $\mathrm{d}_0(\mathfrak{p})$ (respectively $\mathrm{d}_i(\mathfrak{p})$) is the $\mathcal{M}$-clique obtained by replacing the label of the base (respectively i-th edge) of $\mathfrak{p}$ by $\mathbb{1}_{\mathcal{M}}$.

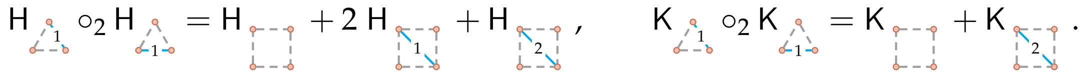
Proposition 1.3.5. *Let $\mathcal{M}$ be a unitary magma. The partial composition of $\mathbf{C}\mathcal{M}$ expresses over the H-basis, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ different from and any valid integer i, as*

$$\mathsf{H}_{\mathfrak{p}} \circ_i \mathsf{H}_{\mathfrak{q}} = \begin{cases} \mathsf{H}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathsf{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathfrak{q}} + \mathsf{H}_{\mathfrak{p} \circ_i \mathrm{d}_0(\mathfrak{q})} + \mathsf{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q})} & \text{if } \mathfrak{p}_i \neq \mathbb{1}_{\mathcal{M}} \text{ and } \mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}, \\ \mathsf{H}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathsf{H}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathfrak{q}} & \text{if } \mathfrak{p}_i \neq \mathbb{1}_{\mathcal{M}}, \\ \mathsf{H}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathsf{H}_{\mathfrak{p} \circ_i \mathrm{d}_0(\mathfrak{q})} & \text{if } \mathfrak{q}_0 \neq \mathbb{1}_{\mathcal{M}}, \\ \mathsf{H}_{\mathfrak{p} \circ_i \mathfrak{q}} & \text{otherwise.} \end{cases} \tag{1.3.2}$$

Proposition 1.3.6. *Let $\mathcal{M}$ be a unitary magma. The partial composition of $\mathbf{C}\mathcal{M}$ expresses over the K-basis, for any $\mathcal{M}$-cliques $\mathfrak{p}$ and $\mathfrak{q}$ different from and any valid integer i, as*

$$\mathsf{K}_{\mathfrak{p}} \circ_i \mathsf{K}_{\mathfrak{q}} = \begin{cases} \mathsf{K}_{\mathfrak{p} \circ_i \mathfrak{q}} & \text{if } \mathfrak{p}_i \star \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}, \\ \mathsf{K}_{\mathfrak{p} \circ_i \mathfrak{q}} + \mathsf{K}_{\mathrm{d}_i(\mathfrak{p}) \circ_i \mathrm{d}_0(\mathfrak{q})} & \text{otherwise.} \end{cases} \tag{1.3.3}$$

For instance, in $\mathbf{C}\mathbb{Z}$,


(1.3.4)

2 Quotients and suboperads

2.1 Operads on subfamilies of $\mathcal{M}$-cliques

We now define quotients of $\mathbf{C}\mathcal{M}$, leading to the construction of some new operads involving various combinatorial objects which are, basically, $\mathcal{M}$-cliques with some restrictions. Figure 1 shows a diagram containing all the considered quotients and suboperads of $\mathbf{C}\mathcal{M}$.

Bubbles. An $\mathcal{M}$-clique is an $\mathcal{M}$-*bubble* if it has no solid diagonals. Let $\mathfrak{R}_{\mathrm{Bub}\mathcal{M}}$ be the subspace of $\mathbf{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques that are not bubbles. As quotient of vector spaces, $\mathrm{Bub}\mathcal{M} := \mathbf{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Bub}\mathcal{M}}}$ is the linear span of all $\mathcal{M}$-bubbles. Moreover, the space $\mathrm{Bub}\mathcal{M}$ is a quotient of operads of $\mathbf{C}\mathcal{M}$. When $\mathcal{M}$ is finite, the dimensions of $\mathrm{Bub}\mathcal{M}$ satisfy, for any $n \geqslant 2$, $\dim \mathrm{Bub}\mathcal{M}(n) = m^{n+1}$, where $m := \#\mathcal{M}$.

White $\mathcal{M}$-cliques. An $\mathcal{M}$-clique is *white* if it has no solid edges nor solid base. Let $\mathrm{Whi}\mathcal{M}$ be the subspace of $\mathrm{C}\mathcal{M}$ of all white $\mathcal{M}$-cliques. The space $\mathrm{Whi}\mathcal{M}$ is a suboperad of $\mathrm{C}\mathcal{M}$. When $\mathcal{M}$ is finite, the dimensions of $\mathrm{Whi}\mathcal{M}$ satisfy, for any $n \geqslant 2$, $\dim \mathrm{Whi}\mathcal{M}(n) = m^{(n+1)(n-2)/2}$, where $m := \#\mathcal{M}$.

Restricting the crossing. The *crossing* of a solid diagonal of an $\mathcal{M}$-clique $\mathfrak{p}$ is the number of solid diagonals crossing it. The *crossing* of $\mathfrak{p}$ is the maximal crossing of its solid diagonals. For any integer $k \geqslant 0$, let $\mathfrak{R}_{\mathrm{Cro}_k\mathcal{M}}$ be the subspace of $\mathrm{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques of crossings greater than k. As quotient of vector spaces, $\mathrm{Cro}_k\mathcal{M} := \mathrm{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Cro}_k\mathcal{M}}}$ is the linear span of all $\mathcal{M}$-cliques of crossings no greater than k. Moreover, the space $\mathrm{Cro}_k\mathcal{M}$ is both a quotient and a suboperad of $\mathrm{C}\mathcal{M}$. Observe that $\mathrm{Cro}_0\mathcal{M}$ is the operad $\mathrm{Bub}\mathcal{M}$. Let us set $\mathrm{NC}\mathcal{M} := \mathrm{Cro}_0\mathcal{M}$. Any $\mathcal{M}$-clique of $\mathrm{NC}\mathcal{M}$ is a noncrossing configuration [6] where each diagonal is decorated by an element of $\mathcal{M} \setminus \{\mathbb{1}_\mathcal{M}\}$. These operads $\mathrm{NC}\mathcal{M}$ have a lot of nice properties and will be studied in Section 3.

Acyclic $\mathcal{M}$-cliques. An $\mathcal{M}$-clique is *acyclic* if it does not contain any cycle formed by solid arcs. Let $\mathfrak{R}_{\mathrm{Acy}\mathcal{M}}$ be the subspace of $\mathrm{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques that are not acyclic. As quotient of vector spaces, $\mathrm{Acy}\mathcal{M} := \mathrm{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Acy}\mathcal{M}}}$ is the linear span of all acyclic $\mathcal{M}$-cliques. When $\mathcal{M}$ has no nontrivial unit divisors, the space $\mathrm{Acy}\mathcal{M}$ is a quotient of operads of $\mathrm{C}\mathcal{M}$. Any $\mathbb{D}_0$-clique of $\mathrm{Acy}\mathbb{D}_0$ can be seen as a forest of trees. The dimensions of this operad begin by 1, 7, 38, 291, 2932 (Sequence **A001858**, except for the first terms).

Nesting-free $\mathcal{M}$-cliques. A solid arc (x', y') is *nested* in a solid arc (x, y) of an $\mathcal{M}$-clique $\mathfrak{p}$ if $x \leqslant x' < y' \leqslant y$. We say that $\mathfrak{p}$ is *nesting-free* if for any solid arcs (x, y) and (x', y') of $\mathfrak{p}$ such that (x', y') is nested in (x, y), $(x', y') = (x, y)$. Let $\mathfrak{R}_{\mathrm{Nes}\mathcal{M}}$ be the subspace of $\mathrm{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques that are not nesting-free. As quotient of vector spaces, $\mathrm{Nes}\mathcal{M} := \mathrm{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Nes}\mathcal{M}}}$ is the linear span of all nesting-free $\mathcal{M}$-cliques. When $\mathcal{M}$ has no nontrivial unit divisors, the space $\mathrm{Nes}\mathcal{M}$ is a quotient of operads of $\mathrm{C}\mathcal{M}$. Any $\mathbb{D}_0$-clique of $\mathrm{Nes}\mathbb{D}_0$ can be seen as an nesting-free clique. The dimensions of this operad begin by 1, 5, 14, 42, 132, and are Catalan numbers (Sequence **A000108**, except for the first terms). In the same way as considering $\mathcal{M}$-cliques of crossings no greater than k leads to quotients $\mathrm{Cro}_k\mathcal{M}$ of $\mathrm{C}\mathcal{M}$, it is possible to define analogous quotients $\mathrm{Nes}_k\mathcal{M}$ spanned by $\mathcal{M}$-cliques having solid arcs that nest at most k other ones.

Restricting the degree. The *degree* of a vertex x of an $\mathcal{M}$-clique $\mathfrak{p}$ is the number of solid arcs adjacent to x. The *degree* of $\mathfrak{p}$ is the maximal degree of its vertices. For any integer $k \geqslant 0$, let $\mathfrak{R}_{\mathrm{Deg}_k\mathcal{M}}$ be the subspace of $\mathrm{C}\mathcal{M}$ generated by all $\mathcal{M}$-cliques of degrees greater than k. As quotient of vector spaces, $\mathrm{Deg}_k\mathcal{M} := \mathrm{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Deg}_k\mathcal{M}}}$ is the linear span of all $\mathcal{M}$-cliques of degrees no greater than k. When $\mathcal{M}$ has no nontrivial unit divisors, the space $\mathrm{Deg}_k\mathcal{M}$ is a quotient of operads of $\mathrm{C}\mathcal{M}$. Observe that $\mathrm{Deg}_0\mathcal{M}$ is the associative operad As. Let us set $\mathrm{Inv}\mathcal{M} := \mathrm{Deg}_1\mathcal{M}$. Any $\mathbb{D}_0$-clique of $\mathrm{Inv}\mathbb{D}_0$ of size n can be seen as a partition of the set $[n+1]$ in singletons or pairs. In this case, $\mathrm{Inv}\mathbb{D}_0$ involves involutions, or equivalently standard Young tableaux. The dimensions of this operad begin by 1, 4, 10, 26, 76 (Sequence **A000085**, except for the first terms). Moreover, the dimensions of $\mathrm{Inv}\mathbb{D}_1$ begin by 1, 7, 25, 81, 331 (Sequence **A047974**, except for the first terms). Besides, any $\mathbb{D}_0$-clique of $\mathrm{Deg}_2\mathbb{D}_0$ can be seen as a *thunderstorm graph* (*i.e.*, a graph where connected components are cycles or paths). The dimensions of this operad begin by 1, 8, 41, 253, 1858 (Sequence **A136281**, except for the first terms).

2.2 Mixing quotients and substructures

For any operad $\mathcal{O}$ and ideals of operads $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of $\mathcal{O}$, the space $\mathfrak{R}_1 + \mathfrak{R}_2$ is still an ideal of operads of $\mathcal{O}$, and $\mathcal{O}/_{\mathfrak{R}_1+\mathfrak{R}_2}$ is a quotient of operads of both $\mathcal{O}/_{\mathfrak{R}_1}$ and $\mathcal{O}/_{\mathfrak{R}_2}$. Moreover, if $\mathcal{O}'$ is a suboperad of $\mathcal{O}$ and $\mathfrak{R}$ is an ideal of operads of $\mathcal{O}$, the space $\mathfrak{R} \cap \mathcal{O}'$ is an ideal of operads of $\mathcal{O}'$, and $\mathcal{O}'/_{\mathfrak{R}\cap\mathcal{O}'}$ is a quotient of operads of $\mathcal{O}'$. For these reasons, we can combine the constructions of Section 2.1 to build a bunch of new quotients of operads of $\mathrm{C}\mathcal{M}$.

When $\mathcal{M}$ is finite and has cardinal 2, several interesting phenomena occur already. In this case, $\mathcal{M}$ is necessarily isomorphic to $\mathbb{N}_2$ or to $\mathbb{D}_0$, but only $\mathbb{D}_0$ satisfies the conditions required by all the propositions of Section 2.1. The obtained substructures of $\mathrm{C}\mathbb{D}_0$ are operads that involve some very classical combinatorial objects. For instance:

Schröder trees. Let $\mathrm{Sch}\mathcal{M} := \mathrm{Whi}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Cro}_0\mathcal{M}}\cap\mathrm{Whi}\mathcal{M}}$. The operad $\mathrm{Sch}\mathbb{D}_0$ involves Schröder trees. Its dimensions begin by 1, 1, 3, 11, 45 (Sequence **A001003**).

Forests of paths. Let $\mathrm{Pat}\mathcal{M} := \mathrm{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Acy}\mathcal{M}}+\mathfrak{R}_{\mathrm{Deg}_2\mathcal{M}}}$. The operad $\mathrm{Pat}\mathbb{D}_0$ involves forests of non-rooted trees that are paths. Its dimensions begin by 1, 7, 34, 206, 1486 (Sequence **A011800**, except for the first terms).

Forests of trees. Let $\mathrm{For}\mathcal{M} := \mathrm{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Acy}\mathcal{M}}+\mathfrak{R}_{\mathrm{Cro}_0\mathcal{M}}}$. The operad $\mathrm{For}\mathbb{D}_0$ involves forests of rooted trees without crossing edges. Its dimensions begin by 1, 7, 33, 181, 1083 (Sequence **A054727**, except for the first terms).

Motzkin configurations. Let $\mathrm{Mot}\mathcal{M} := \mathrm{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Cro}_0\mathcal{M}}+\mathfrak{R}_{\mathrm{Deg}_1\mathcal{M}}}$. The operad $\mathrm{Mot}\mathbb{D}_0$ involves Motzkin paths. Its dimensions begin by 1, 4, 9, 21, 51 (Sequence **A001006**, except for the first terms).

Dissections of polygons. Let $\mathrm{Dis}\mathcal{M} := \mathrm{Whi}\mathcal{M}/_{(\mathfrak{R}_{\mathrm{Cro}_0\mathcal{M}}+\mathfrak{R}_{\mathrm{Deg}_1\mathcal{M}})\cap\mathrm{Whi}\mathcal{M}}$. The operad $\mathrm{Dis}\mathbb{D}_0$ involves dissections of polygons by strictly disjoint diagonals. Its dimensions begin by 1, 1, 3, 6, 13 (Sequence **A093128**, except for the first terms).

Lucas configurations. Let $\mathsf{Luc}\mathcal{M} := \mathrm{C}\mathcal{M}/_{\mathfrak{R}_{\mathrm{Bub}\mathcal{M}}+\mathfrak{R}_{\mathrm{Deg}_1\mathcal{M}}}$. The operad $\mathsf{Luc}\mathbb{D}_0$ involves $\mathbb{D}_0$-bubbles $\mathfrak{p}$ such that any vertex of $\mathfrak{p}$ belongs to at most one solid edge. Its dimensions begin by 1, 4, 7, 11, 18, and are Lucas numbers (Sequence **A000032**, except for the first terms).

2.3 Constructing existing operads

We give here three examples of already known operads that can be build through the construction C.

Bicolored noncrossing configurations. The *operad of bicolored noncrossing configurations* BNC is an operad defined in [4] which involves noncrossing configurations where each solid diagonal can be blue or red, and each edge can be blue or uncolored. This operad is in fact a special case of our general construction C. Let $\mathcal{M}_{\mathsf{BNC}} := \{\mathbb{1}, \mathtt{a}, \mathtt{b}\}$ be the unitary magma wherein operation $\star$ is defined so that $\mathtt{a}$ and $\mathtt{b}$ are idempotent, and $\mathtt{a} \star \mathtt{b} = \mathbb{1} = \mathtt{b} \star \mathtt{a}$. Observe that $\mathcal{M}_{\mathsf{BNC}}$ is a commutative unitary magma, but, since $(\mathtt{b} \star \mathtt{a}) \star \mathtt{a} = \mathtt{a}$ and $\mathtt{b} \star (\mathtt{a} \star \mathtt{a}) = \mathbb{1}$, the operation $\star$ is not associative.

Proposition 2.3.1. *The suboperad of* $\mathrm{NC}\mathcal{M}_{\mathsf{BNC}}$ *consisting in its unit and all* $\mathcal{M}_{\mathsf{BNC}}$*-noncrossing configurations without edges labeled by* $\mathbb{1}$ *is isomorphic to the operad* BNC*.*

Multi-tildes and double multi-tildes. Appearing from the context of formal languages theory, *multi-tildes* are operators introduced in [2] as tools offering a convenient way to express regular languages. As shown in [11], the set of all multi-tildes admits a very natural structure of an operad MT. *Double multi-tildes* are extensions of these operators introduced in [8] that increase their expressiveness and admit also a structure of an operad DMT. Let $\mathcal{M}_{\mathsf{DMT}}$ be the unitary magma $\mathcal{M}_{\mathsf{DMT}} := \mathbb{D}_0^2$ and $\mathcal{M}_{\mathsf{MT}}$ be the sub-unitary magma of $\mathcal{M}_{\mathsf{DMT}}$ on the set $\{(\mathbb{1}, \mathbb{1}), (0, \mathbb{1})\}$.

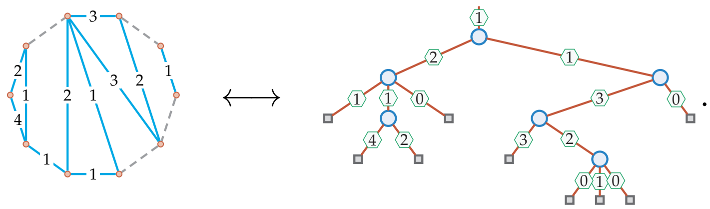
Proposition 2.3.2. *The operad* $\mathrm{C}\mathcal{M}_{\mathsf{DMT}}$ *(respectively* $\mathrm{C}\mathcal{M}_{\mathsf{MT}}$*) is isomorphic to the suboperad of* DMT *(respectively* MT*) consisting in all double (respectively simple) multi-tildes except the three (respectively one) nontrivial ones of arity* 1*.*

3 Operads of decorated noncrossing configurations

In this section, we study in details the suboperad $\mathrm{NC}\mathcal{M}$ of $\mathrm{C}\mathcal{M}$. As observed in Section 2.1, this operad involves all $\mathcal{M}$-cliques that do not admit crossing solid diagonals. We call *$\mathcal{M}$-noncrossing configurations* such objects.

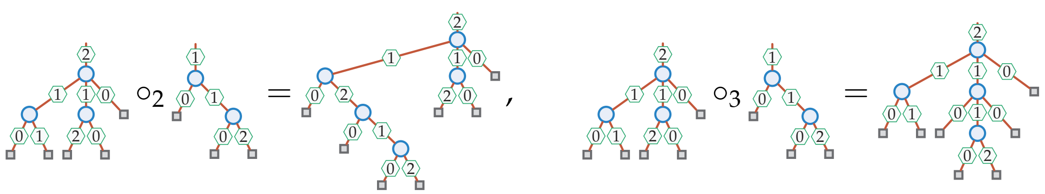
3.1 General properties

The set of all $\mathcal{M}$-noncrossing configurations is in one-to-one correspondence with the set of Schröder trees (*i.e.*, rooted planar trees where internal nodes have arities 2 or more) where the edges adjacent to the roots are labeled on $\mathcal{M}$, the edges connecting two internal nodes are labeled on $\mathcal{M} \setminus \{\mathbb{1}_{\mathcal{M}}\}$, and the edges adjacent to the leaves are labeled on $\mathcal{M}$. This is realized by computing the dual trees of $\mathcal{M}$-noncrossing configurations by considering the labels of the solid diagonals. We call these trees $\mathcal{M}$-*dual trees*. Here is an example of a $\mathbb{Z}$-noncrossing configuration and the $\mathbb{Z}$-dual tree encoding it:



(3.1.1)

By seeing the elements of $\mathrm{NC}\mathcal{M}$ as $\mathcal{M}$-dual trees, we can rephrase the partial composition of this operad as follows. If $\mathfrak{s}$ and $\mathfrak{t}$ are two $\mathcal{M}$-dual trees and i is a valid integer, the tree $\mathfrak{s} \circ_i \mathfrak{t}$ is computed by grafting the root of $\mathfrak{t}$ to the i-th leaf of $\mathfrak{s}$. Then, by denoting by b the label of the edge adjacent to the root of $\mathfrak{t}$ and by a the label of the edge adjacent to the i-th leaf of $\mathfrak{s}$, we have two cases to consider, depending on the value of $c := a \star b$. If $c \neq \mathbb{1}_{\mathcal{M}}$, we label the edge connecting $\mathfrak{s}$ and $\mathfrak{t}$ by c. Otherwise, when $c = \mathbb{1}_{\mathcal{M}}$, we contract the edge connecting $\mathfrak{s}$ and $\mathfrak{t}$ by merging the root of $\mathfrak{t}$ and the father of the i-th leaf of $\mathfrak{s}$. For instance, in $\mathrm{NC}\mathbb{N}_3$, we have



(3.1.2)

Let $\mathcal{T}_{\mathcal{M}}$ be the set of all $\mathcal{M}$-cliques of arity 2. We call such cliques $\mathcal{M}$-*triangles*.

Proposition 3.1.1. *Let $\mathcal{M}$ be a unitary magma. The set $\mathcal{T}_{\mathcal{M}}$ is the unique minimal generating set of* $\mathrm{NC}\mathcal{M}$.

Proposition 3.1.2. *Let $\mathcal{M}$ be a finite unitary magma and m be its cardinality. The Hilbert series $\mathcal{H}_{\mathrm{NC}\mathcal{M}}(t)$ of* $\mathrm{NC}\mathcal{M}$ *satisfies*

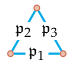
$$t + \left(m^3 - 2m^2 + 2m - 1\right) t^2 + \left(2m^2 t - 3mt + 2t - 1\right) \mathcal{H}_{\mathrm{NC}\mathcal{M}}(t) + (m-1)\, \mathcal{H}_{\mathrm{NC}\mathcal{M}}(t)^2 = 0. \tag{3.1.3}$$

From this result, together with classical arguments involving Narayana numbers, we obtain that for all $n \geqslant 2$,

$$\dim \mathrm{NC}\mathcal{M}(n) = \sum_{0 \leqslant k \leqslant n-2} m^{n+k+1} (m-1)^{n-k-2} \frac{1}{k+1} \binom{n-2}{k} \binom{n-1}{k}. \tag{3.1.4}$$

For instance, when $\#\mathcal{M} = 2$, the dimensions of $\mathrm{NC}\mathcal{M}$ begin by 1, 8, 48, 352, 2880 (Sequence **A054726**, except for the first terms).

3.2 Presentation, Koszulity, and Koszul dual

In what follows, $\mathcal{M}$-triangles $\mathfrak{p} =$ 

are denoted by words $\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \in \mathcal{M}^3$.

Theorem 3.2.1. *Let $\mathcal{M}$ be a finite unitary magma. The operad $\mathrm{NC}\mathcal{M}$ is binary, quadratic, Koszul, and admits the presentation $\mathrm{NC}\mathcal{M} \simeq \mathrm{Free}(\mathcal{T}_{\mathcal{M}})/_{\langle \mathfrak{R} \rangle}$, where $\mathfrak{R}$ is the space of relations generated by*

$$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \circ_1 \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 - \mathfrak{p}_1\mathfrak{r}_2\mathfrak{p}_3 \circ_1 \mathfrak{r}_1\mathfrak{q}_2\mathfrak{q}_3, \qquad \text{if } \mathfrak{p}_2 \star \mathfrak{q}_1 = \mathfrak{r}_2 \star \mathfrak{r}_1 \neq \mathbb{1}_{\mathcal{M}}, \tag{3.2.1a}$$

$$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \circ_1 \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 - \mathfrak{p}_1\mathfrak{q}_2\mathfrak{r}_3 \circ_2 \mathfrak{r}_1\mathfrak{q}_3\mathfrak{p}_3, \qquad \text{if } \mathfrak{p}_2 \star \mathfrak{q}_1 = \mathfrak{r}_3 \star \mathfrak{r}_1 = \mathbb{1}_{\mathcal{M}}, \tag{3.2.1b}$$

$$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \circ_2 \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 - \mathfrak{p}_1\mathfrak{p}_2\mathfrak{r}_3 \circ_2 \mathfrak{r}_1\mathfrak{q}_2\mathfrak{q}_3, \qquad \text{if } \mathfrak{p}_3 \star \mathfrak{q}_1 = \mathfrak{r}_3 \star \mathfrak{r}_1 \neq \mathbb{1}_{\mathcal{M}}, \tag{3.2.1c}$$

where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3 \in \mathcal{M}$.

Proof. The proof is long, technical, but classical and uses techniques from rewriting theory [1]. It consists in defining a rewrite rule $\to$ from $\mathfrak{R}$ on syntax trees of $\mathcal{M}$-triangles, by showing that $\to$ is convergent, and prove that $\to$ admits as many normal forms as basis elements of $\mathrm{NC}\mathcal{M}$ of arity n for all $n \geqslant 1$. The fact that $\mathrm{NC}\mathcal{M}$ is Koszul is a consequence of the existence of such a rewrite rule $\to$ (see [5, 9]). $\square$

We can now compute a presentation of the Koszul dual $\mathrm{NC}\mathcal{M}^!$ of $\mathrm{NC}\mathcal{M}$ from the presentation of $\mathrm{NC}\mathcal{M}$ provided by Theorem 3.2.1.

Proposition 3.2.2. *Let $\mathcal{M}$ be a finite unitary magma. The operad $\mathrm{NC}\mathcal{M}^!$ admits the presentation $\mathrm{NC}\mathcal{M}^! \simeq \mathrm{Free}(\mathcal{T}_{\mathcal{M}})/_{\langle \mathfrak{R}^\perp \rangle}$, where $\mathfrak{R}^\perp$ is the space of relations generated by*

$$\sum_{\substack{\mathfrak{p}_2, \mathfrak{q}_1 \in \mathcal{M}, \mathfrak{p}_2 \star \mathfrak{q}_1 = \delta}} \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \circ_1 \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3, \qquad \text{where } \mathfrak{p}_1, \mathfrak{p}_3, \mathfrak{q}_2, \mathfrak{q}_3 \in \mathcal{M}, \delta \in \mathcal{M} \setminus \{\mathbb{1}_{\mathcal{M}}\}, \tag{3.2.2a}$$

$$\sum_{\substack{\mathfrak{p}_2, \mathfrak{q}_1 \in \mathcal{M}, \mathfrak{p}_2 \star \mathfrak{q}_1 = \mathbb{1}_{\mathcal{M}}}} \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \circ_1 \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 - \mathfrak{p}_1\mathfrak{q}_2\mathfrak{p}_2 \circ_2 \mathfrak{q}_1\mathfrak{q}_3\mathfrak{p}_3, \qquad \text{where } \mathfrak{p}_1, \mathfrak{p}_3, \mathfrak{q}_2, \mathfrak{q}_3 \in \mathcal{M}, \tag{3.2.2b}$$

$$\sum_{\substack{\mathfrak{p}_3, \mathfrak{q}_1 \in \mathcal{M}, \mathfrak{p}_3 \star \mathfrak{q}_1 = \delta}} \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \circ_2 \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3, \qquad \text{where } \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{q}_3 \in \mathcal{M}, \delta \in \mathcal{M} \setminus \{\mathbb{1}_{\mathcal{M}}\}. \tag{3.2.2c}$$

Proposition 3.2.3. *Let $\mathcal{M}$ be a finite unitary magma and m be its cardinality. The Hilbert series $\mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t)$ of $\mathrm{NC}\mathcal{M}^!$ satisfies*

$$t + (m-1)t^2 + \left(2m^2t - 3mt + 2t - 1\right) \mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t) + \left(m^3 - 2m^2 + 2m - 1\right) \mathcal{H}_{\mathrm{NC}\mathcal{M}^!}(t)^2 = 0. \tag{3.2.3}$$

This is the generating series of all noncrossing configurations where all edges and bases are labeled by pairs $(a, a) \in \mathcal{M}^2$, and all solid diagonals are labeled by pairs $(a, b) \in \mathcal{M}^2$ where $a \neq b$.

Proposition 3.2.3 hence provides a combinatorial description of the elements of $\mathrm{NC}\mathcal{M}^!$. For instance, when $\#\mathcal{M} = 2$, the dimensions of $\mathrm{NC}\mathcal{M}^!$ begin by 1, 8, 80, 992, 13760 (Sequence **A234596**). It is worthwhile to observe that the dimensions of $\mathrm{NC}\mathcal{M}^!$ in this case are the ones of the operad BNC [4] (see Section 2.3).

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